Generalized Infinite-Dimensional Weighted Non-Linear Homotopy-Invariant Boundary and Coboundary Maps

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Generalized Infinite-Dimensional Weighted Non-Linear Homotopy-Invariant Boundary Map

Definition 1 (Infinarray). *An **infinarray** is a data structure denoted as A, where each element $A_{i,j,k,\ldots}$ corresponds to an entry in an array extending infinitely in multiple dimensions.*

Definition 2 (Generalized Infinite-Dimensional Weighted Non-Linear Homotopy-Invariant Boundary Map). *A **generalized infinite-dimensional weighted non-linear homotopy-invariant boundary map** ∂_g is an advanced boundary operator acting on infinarrays, incorporating infinite-dimensional, weighted, non-linear, and homotopy-invariant properties to handle the complexities of (co)homology theories.*

- *Acts on an infinarray A such that:*

$$\partial_g A_{i,j,k,\ldots} = \sum_d w_d (-1)^d f(A_{i_1,\ldots,i_d-1,\ldots,i_n})$$

- *Where:*
  - w_d *are weights assigned to each dimension* d*.*
  - $(-1)^d$ *introduces sign changes based on the dimension* d*.*
  - f *is a non-linear function applied to each element.*
  - *The map is **homotopy-invariant**, meaning* $\partial_g h(A) = h(\partial_g A)$ *for a homotopy* h*.*

Properties of the Boundary Map

Property 1 (Linearity). *The boundary map is linear:*

$$\partial_g (c_1 A + c_2 B) = c_1 \partial_g A + c_2 \partial_g B$$

for any constants c_1 and c_2.

Property 2 (Chain Rule)**.** *The boundary map satisfies the chain rule:*

$$\partial_g(\partial_g A) = 0$$

ensuring that applying the boundary map twice yields zero.

Property 3 (Weight Invariance)**.** *The weights w_d can be chosen to reflect the significance of different dimensions, allowing flexibility in the representation of the infinarray.*

Property 4 (Non-Linearity)**.** *The non-linear function f can introduce complex interactions between elements of the infinarray, capturing higher-order relationships.*

Property 5 (Homotopy Invariance)**.** *The boundary map remains invariant under homotopy transformations, preserving topological properties:*

$$\partial_g h(A) = h(\partial_g A)$$

for a homotopy h.

Property 6 (Commutativity with Other Operators)**.** *The boundary map commutes with other linear operators T:*

$$T(\partial_g A) = \partial_g(TA)$$

for any linear operator T.

Property 7 (Stability Under Transformations)**.** *The boundary map is stable under transformations that preserve the infinarray structure:*

$$\partial_g(T(A)) = T(\partial_g A)$$

where T is a transformation preserving the infinarray structure.

Generalized Infinite-Dimensional Weighted Non-Linear Homotopy-Invariant Coboundary Map

Definition 3 (Generalized Infinite-Dimensional Weighted Non-Linear Homotopy-Invariant Coboundary Map)**.** *A* ***generalized infinite-dimensional weighted non-linear homotopy-invariant coboundary map*** *δ_g is an advanced coboundary operator acting on infinarrays, incorporating infinite-dimensional, weighted, non-linear, and homotopy-invariant properties to handle the complexities of cohomology theories.*

- *Acts on an infinarray B such that:*

$$\delta_g B_{i,j,k,\ldots} = \sum_d w_d(-1)^d f(B_{i_1,\ldots,i_d+1,\ldots,i_n})$$

- *Where:*
  - w_d *are weights assigned to each dimension* d.
  - $(-1)^d$ *introduces sign changes based on the dimension* d.
  - f *is a non-linear function applied to each element.*
  - *The map is* ***homotopy-invariant****, meaning* $\delta_g h(B) = h(\delta_g B)$ *for a homotopy* h.

Properties of the Coboundary Map

Property 8 (Linearity)**.** *The coboundary map is linear:*

$$\delta_g(c_1 B + c_2 C) = c_1 \delta_g B + c_2 \delta_g C$$

for any constants c_1 *and* c_2.

Property 9 (Chain Rule)**.** *The coboundary map satisfies the chain rule:*

$$\delta_g(\delta_g B) = 0$$

ensuring that applying the coboundary map twice yields zero.

Property 10 (Weight Invariance)**.** *The weights* w_d *can be chosen to reflect the significance of different dimensions, allowing flexibility in the representation of the infinarray.*

Property 11 (Non-Linearity)**.** *The non-linear function* f *can introduce complex interactions between elements of the infinarray, capturing higher-order relationships.*

Property 12 (Homotopy Invariance)**.** *The coboundary map remains invariant under homotopy transformations, preserving topological properties:*

$$\delta_g h(B) = h(\delta_g B)$$

for a homotopy h.

Property 13 (Commutativity with Other Operators)**.** *The coboundary map commutes with other linear operators* T*:*

$$T(\delta_g B) = \delta_g(TB)$$

for any linear operator T.

Property 14 (Stability Under Transformations)**.** *The coboundary map is stable under transformations that preserve the infinarray structure:*

$$\delta_g(T(B)) = T(\delta_g B)$$

where T *is a transformation preserving the infinarray structure.*

Examples

Example 1 (Singular Homology). *Consider a topological space X with an infinite covering. An infinarray A can represent the chain complex where each entry corresponds to a singular simplex in the covering.*

$$\partial_g A_{i,j} = w_1(-1)^1 f(A_{i-1,j}) + w_2(-1)^2 f(A_{i,j-1})$$

Where:

- $f(x) = x^2$ *is a non-linear function.*
- w_1 *and* w_2 *are weights assigned to the dimensions.*

Example 2 (Cech Cohomology). *For a topological space Y, an infinarray B represents the cochain complex where each entry corresponds to a Cech cochain.*

$$\delta_g B_{i,j} = w_1(-1)^1 f(B_{i+1,j}) + w_2(-1)^2 f(B_{i,j+1})$$

Where:

- $f(x) = \sin(x)$ *is a non-linear function.*
- w_1 *and* w_2 *are weights assigned to the dimensions.*

Further Properties of Boundary and Coboundary Maps

Property 15 (Adjointness). *In certain contexts, the boundary map ∂_g and the coboundary map δ_g can be adjoints of each other:*

$$\langle \partial_g A, B \rangle = \langle A, \delta_g B \rangle$$

for appropriate inner products $\langle \cdot, \cdot \rangle$.

Property 16 (Spectral Sequence Compatibility). *The boundary and coboundary maps are compatible with spectral sequences, allowing for the filtration and analysis of complex structures:*

$$\partial_g E_r^{p,q} = E_{r-1}^{p,q+1} \quad \text{and} \quad \delta_g E_r^{p,q} = E_{r+1}^{p,q-1}$$

where $E_r^{p,q}$ denotes the r-th page of a spectral sequence.

Property 17 (Homological Perturbation Lemma Compatibility). *The boundary and coboundary maps are compatible with the homological perturbation lemma, enabling the study of deformations and perturbations in complex structures:*

$$\partial'_g = \partial_g + \delta_g h \partial_g$$

where h is a homotopy operator.

Property 18 (Module Structure)**.** *The boundary and coboundary maps can act on modules over a ring R, extending their applicability:*

$$\partial_g(r \cdot A) = r \cdot \partial_g A$$

and

$$\delta_g(r \cdot B) = r \cdot \delta_g B$$

for any $r \in R$.

Property 19 (Cup Product Compatibility)**.** *The coboundary map is compatible with the cup product in cohomology, allowing for the study of product structures:*

$$\delta_g(B \smile C) = (\delta_g B) \smile C + (-1)^{|B|} B \smile (\delta_g C)$$

where $\smile$ denotes the cup product.

Additional Examples

Example 3 (Morse Homology)**.** *Consider a smooth manifold M with an infinite sequence of critical points. An infinarray A can represent the chain complex where each entry corresponds to a critical point.*

$$\partial_g A_{i,j} = w_1(-1)^1 f(A_{i-1,j}) + w_2(-1)^2 f(A_{i,j-1})$$

Where:

- *$f(x) = e^x$ is a non-linear function.*
- *w_1 and w_2 are weights assigned to the dimensions.*

Example 4 (De Rham Cohomology)**.** *For a differentiable manifold N, an infinarray B represents the cochain complex where each entry corresponds to a differential form.*

$$\delta_g B_{i,j} = w_1(-1)^1 f(B_{i+1,j}) + w_2(-1)^2 f(B_{i,j+1})$$

Where:

- *$f(x) = \log(x + 1)$ is a non-linear function.*
- *w_1 and w_2 are weights assigned to the dimensions.*

Example 5 (Persistent Homology)**.** *In topological data analysis, persistent homology studies the changes in homology classes as a function of a parameter. An infinarray A can represent the chain complex over an infinite range of parameters.*

$$\partial_g A_{i,j,k} = w_1(-1)^1 f(A_{i-1,j,k}) + w_2(-1)^2 f(A_{i,j-1,k}) + w_3(-1)^3 f(A_{i,j,k-1})$$

Where:

- *$f(x) = \sqrt{x}$ is a non-linear function.*
- *w_1, w_2, and w_3 are weights assigned to the dimensions.*